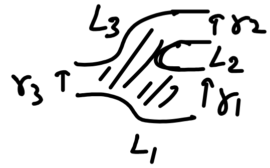
(... definition of wrapped Fukaya category ...)

- Operations: Floer product $HF^*(L_1,L_2)\otimes HF^*(L_2,L_3)\to HF^*(L_1,L_3)$

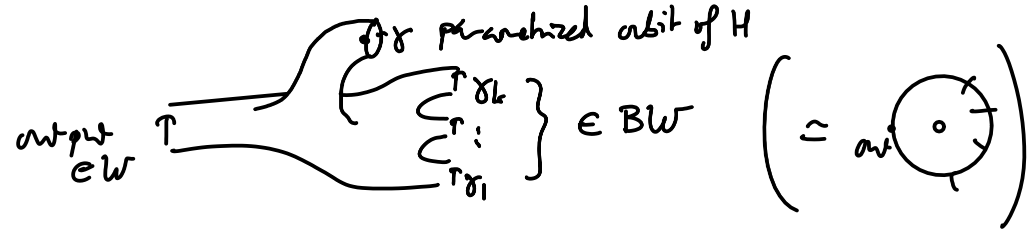
Counts 

cohom. unit 

$\to e_L \in HF^*(L,L)$

- affine varieties have "many" Lagns. in sense below; also holds for:
  Ex: • Lefschetz thimbles in L-fibrations
  • cotangent fiber in T^*N

- There are maps $HH_*(W,W)\xrightarrow{OC} SH^*(M)\xrightarrow{CO} HH^*(W,W)$

CO is count of 

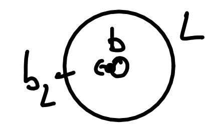
Thm (A.–Seidel): The map OC is an iso. if M is an affine variety.

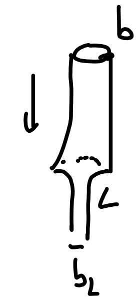
Thm (Ganatra): The map CO is also an iso.

Assume L is compact and $HF^*(L,L)\simeq H^*(L)$

given $b\in SH^*(M)$, can define $b_{|L}\in H^*(L)=HF^*(L,L)$

by b = restriction of $CO(b)\in HH^*(W)$ to L ie.

count holom. punctured discs 



Now, say $b\in SH^1$ ($\leftrightarrow$ holom. vf. on mirror)

Def: (Seidel–Solomon)
L is b-invariant if $[b_{|L}]=0\in H^*(L)$.

Define an <u>equivariant</u> object to be a pair (L, c) where $\partial c = b_{|L}$ at chain level.

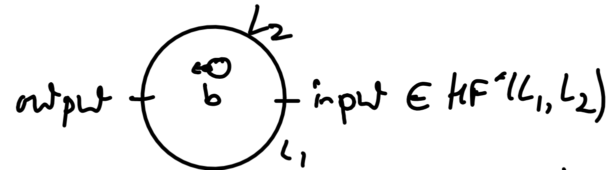
By analogy w/ mirror, the morphisms b/w equivariant objects split according to <u>weights</u> of the action. Hence, want:

L_1, L_2 equivariant $\Rightarrow$ $HF^*(L_1, L_2) = \bigoplus_{\lambda \in \mathbb{K}} HF^*_\lambda(L_1, L_2)$

$\mathbb{K}$ alg. closed

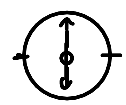
eigenvalues of an operator b^1

$b^1 : HF^*(L_1, L_2) \circlearrowleft$ of degree 0 is defined by count of punctured discs



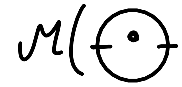
where position of interior puncture $\sqcap^b$ is allowed to vary

There's an interval of such domains

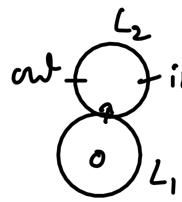


hence get a map of deg 0, not 1.

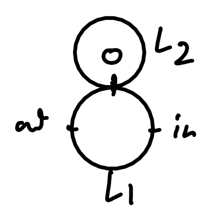
(count index -1 discs)

This does define a cohomology operation even though $\mathcal{M}($  $)$ is an interval ie. has nontrivial boundary.

Indeed, $\partial\mathcal{M} =$



$\cup$



ie. $\mu^2(\cdot, b_{|L_1}) + \mu^2(b_{|L_2}, \cdot)$

Since assume $[b_{|L_i}] = 0$ we get a well-defined cohomology level operation.

* If b is mirror to a dilation of holom. vol. form on a CY, then the weight on $HF^n(L, L)$ is necessarily 1

(while weight on $HF^0(L, L)$ is always 0).

* Note that S^1 acts on $\mathcal{L}M$ by rotating loops. This induces a BV-operator

$$\Delta : SH^*(M) \to SH^{*-1}(M)$$

<u>Claim:</u> The mirror to a holom. dilation is a class $b \in SH^1(M)$ satisfying $\Delta b = e \in SH^0(M)$
(= unit in $SH^0 = HH^*(W, W)$).

Indeed: b holom. vect. field s.t. $L_b \Omega = \Omega$ (holom dilation)

$\Rightarrow$ by Cartan formula, $\Omega = d(\iota_b \Omega)$

Now, $b \in H^*(X, \Lambda^\bullet T) \xrightarrow{\iota.\Omega} H^*(X, \Omega^\bullet)$ isom. since Ω volume form

$\circlearrowright \Delta$ $\quad$ $\circlearrowright$ de Rham diff

(Note $H^*(X, \Omega^\bullet) \simeq HH_*(\mathcal{A}_X, \mathcal{A}_X)$
$\mathcal{A}_X$ dg-model for $Coh(X)$

$\simeq \mathcal{A}_X \otimes_{\mathcal{A}_X} \mathcal{A}_X \otimes_{\mathcal{A}_X}$ diagonal bimodule cyclic tensor product
(using: $\mathcal{A}_X \otimes_{\mathcal{A}_X} \mathcal{A}_X \simeq \mathcal{A}_X$)
$\circlearrowright \mathbb{Z}/2$

$\simeq \otimes \mathcal{A}_X \otimes_{\mathcal{A}_X} \otimes \cdots \otimes \mathcal{A}_X$ (k copies)
$\circlearrowright \mathbb{Z}/_{k\mathbb{Z}}$ by cyclic permutation

$\rightsquigarrow$ an S^1-action induces an operator $\Delta \circlearrowright HH_*$ related to this cyclic perm.)

Now, $\Omega = d(\iota_b \Omega)$
$\quad \updownarrow \quad \updownarrow \quad \updownarrow$
$1 = \Delta(b)$

Let $L_M \subset M \times M$ diagonal.

Given $L_1, L_2 \subset M \times M \rightsquigarrow$ composition $L_1 \circ L_2 \subset M \times M$
(= fiber product over diagonal in middle factors)
$L_1 \times L_2 \subset M \times \underbrace{M \times M} \times M$

Then $SH^*(M) \simeq HF^*(L_M, L_M) \simeq HF^*(\underbrace{L_M \circ L_M \circ \dots, L_M \circ \dots \circ L_M}_{\text{total } k})$ $\Rightarrow \mathbb{Z}/k$-action

Taking inverse limit over $k \to \infty$ of these $\mathbb{Z}/k$-actions,
obtain S^1-action on $SH^*(M)$; the resulting operator agrees w/ Δ.

Outcome: The compatibility of d (De Rham operator) on X
and Δ (BV-op. in SH^*) on M
follows from 1) mirror symmetry for X and M
2) Fourier-Mukai calculus on the two sides
(on sympl. side, formalism is due to
Mau-Wehrheim-Woodward)

This theory has applications to:

1) Lagrangian embedding (Seidel-Solomon, Seidel)
2) (A-Smith) symplectic Khovanov homology
(prove formality of Fukaya cat.)

Expect also to be useful to study Fukaya cats. of var's. coming from
repn theoretic constructions.